# Process Monitoring: Control and Decision

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# motivation

- concerned with Quality of Service
- define effective performance criterion and metrics
- base decisions on measured performance
- establish the measurement and analysis of the quality of service of systems.
- relate quantitative measurements of system QoS with qualitative data on system management processes

# objectives

- **develop methods and tools for system management that**
  – inform system management decision making
  – reduce management errors
  – improve the overall dependability of a managed system.
- **To investigate system measurements**
  – as a basis for automatic, dynamic system re-configurations
  – improve performance and dependability.

# the model

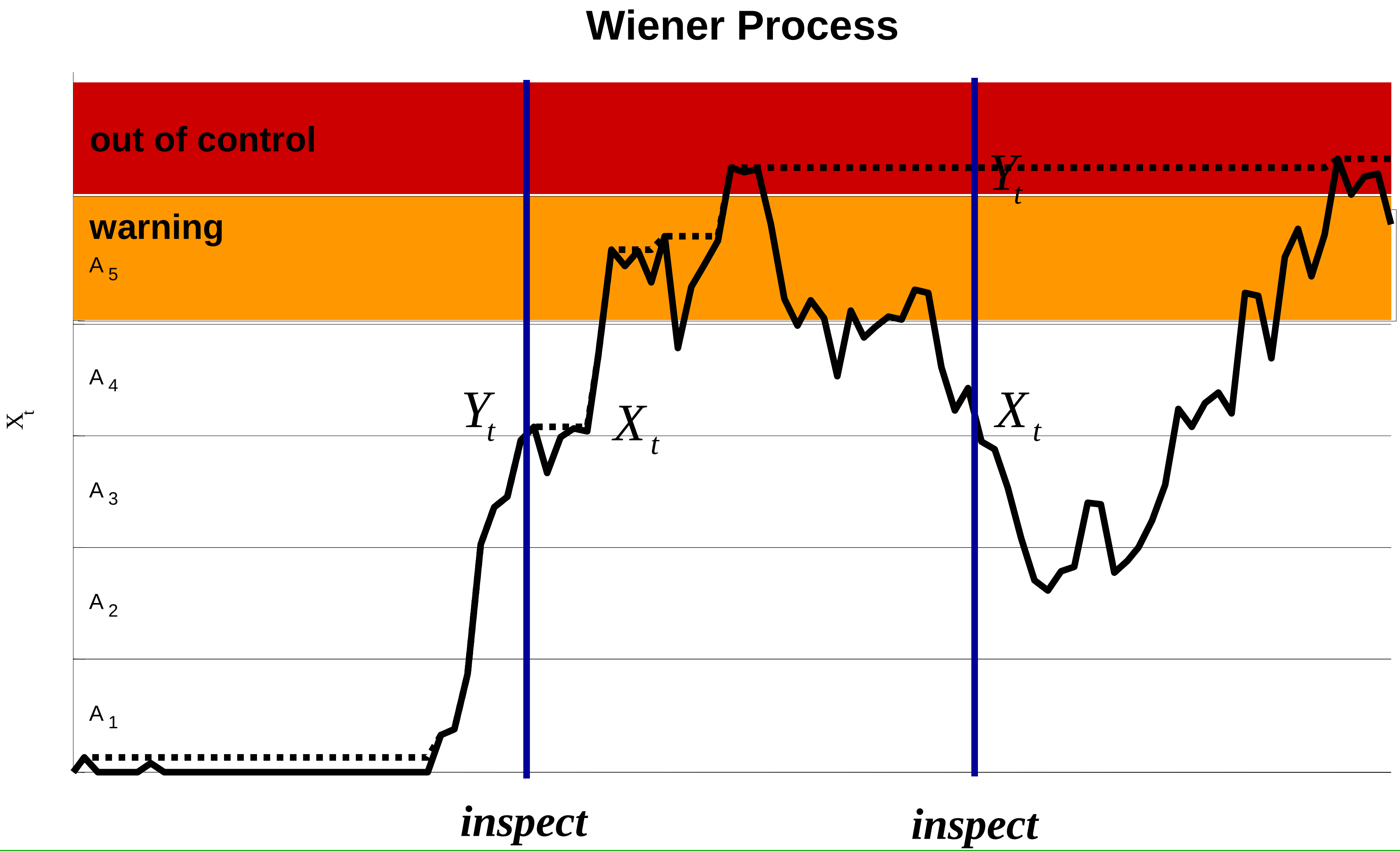


- what is the optimal time between inspections?
- how great should the intervention be?
- system may be a simple or complex
- process state described stochastic process $X_t$
- use decision variable $Y_t$
- may be given or may be constructed $Y_t = A(X_t)$

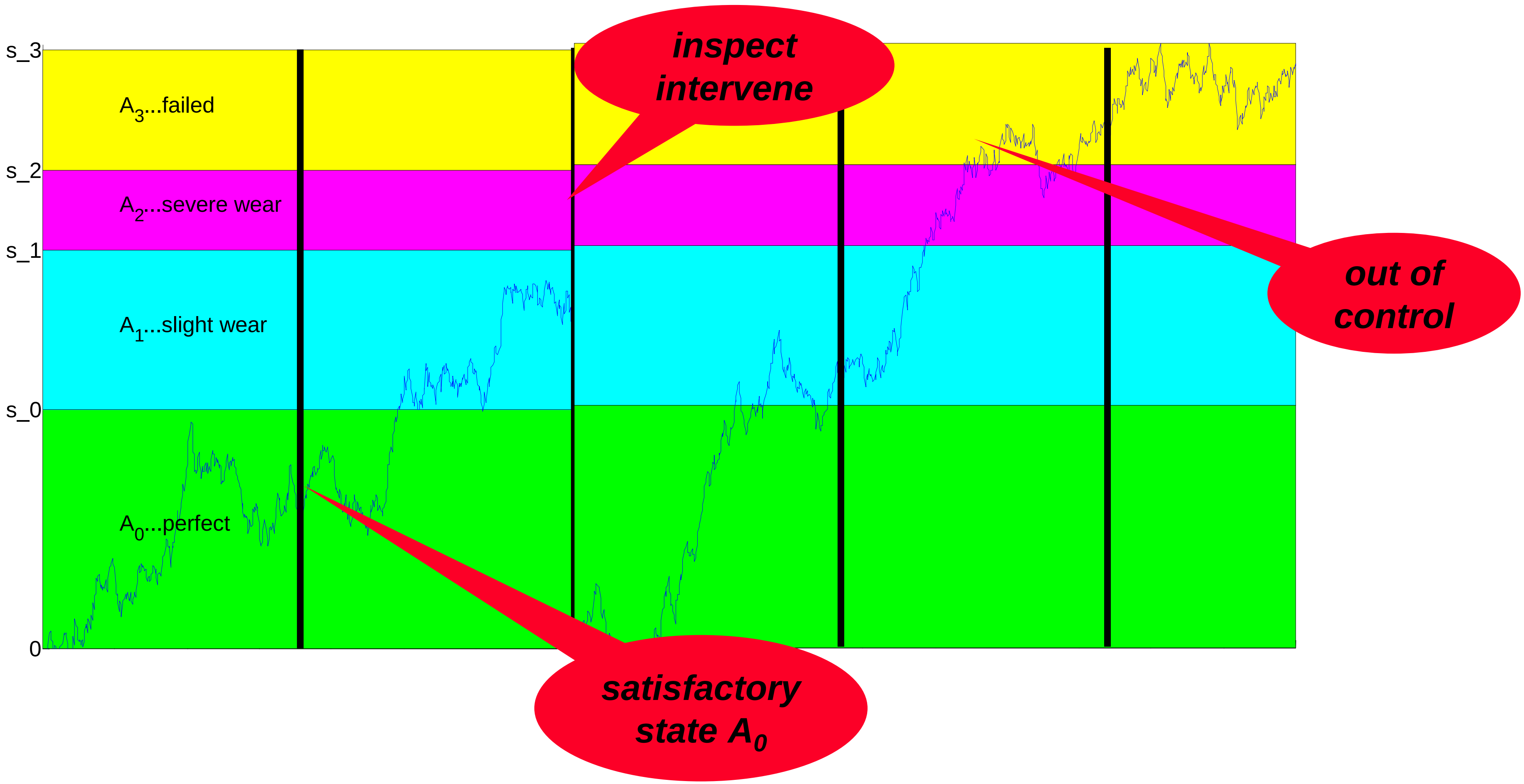
# decision variable

- **The system is inspected to determine its state**
- **Decisions driven by excursions of** $(X_t, Y_t)$
  - **The maximum process** $Y_t = \sup_{0 \le s \le t} X_s$
  - **multivariate process** $Y_t = \|X_t\|$ **(Bessel process)**
  - **CUSUM** $Y_t = \int_{0 \le s \le t} X_s \, ds$ **(Kolmogorov diffusion);**
  - **errors in measurement** $Y_t = u(X_t, \varepsilon)$
  - **covariate processes** $F\left(X_t | Y_t\right)$

# inspection


Wiener Process
out of control
warning
A 5
A 4
A 3
A 2
A 1
X t
Y t
X t
Y t
X t
inspect
inspect

# process example



# definitions

- **The process** $\left(X_t, Y_t\right) \in \Omega \times \mathbb{R}$
- **partitions are and** $\Omega = \bigcup_{j=1..m} A_j \qquad R = \bigcup_{i=1..n} B_i$
- **inspection reveals the state** $(X_t, Y_t) \in A_i \times B_j$
- **action chosen on basis of set** $A_i \times B_j$
- **example: perfect $A_0$; working $A_i$ ; out of control $A^*$**
- **perfect $B_0$; working $B_i$; out of control $B^*$**
- **Hitting times** $T^{u,v} = \inf\left\{t \mid (X_t, Y_t) \in A^\star \times B^\star\right\} \qquad T^{u,v} \sim G^{u,v}(t)$

# Inspection Policy

- decision maker inspects following policy $\pi$
- policy is $\pi = \{\tau_1, \tau_2, \ldots, \tau_n\}$
- inspection is perfect
- actions are instantaneous
- actions are determined by the system state $(X_t, Y_t)$
- state $(X_{t^-}, Y_{t^-}) = (x, y)$ restored to $(X_{t^+}, Y_{t^+}) = (x', y')$
- restoration function $(x', y') = d(x, y)$

# cost equations

## Cost for a single cycle

$$v_\tau(x,y) = c(d(x,y)) + \left\{ v_\tau(0,0) + C_F \right\} p_F^{d(x,y)} + \int_{A_0} \int_{B_0} v_\tau(u,w)\, f_\tau^{d(x,y)}(u,w)\, du\, dw$$

## with

$$p_F^{u,v} = G^{u,v}(t)$$

# optimal policy

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**without discounting**

$$v_\tau(x,y) = \inf_{\tau>0}\left\{c(d(x,y)) + \{v_\tau(0,0) + C_F\}\, p_F^{d(x,y)} + \int_{A_0} v_\tau(u,w)\, f_\tau^{d(x,y)}(u)\, du dw\right\}$$

**Discounted cost solution**

$$v_\tau(x,y) = \inf_{\tau>0}\left\{e^{-r\tau} c(d(x,y)) + \{v_\tau(0,0) + C_F\}\, p_{r,F}^{d(x,y)} + \int_{A_0} e^{-ru} v_\tau(u,w)\, f_\tau^{d(x,y)}(u)\, du dw\right\}$$

**where**

$$p_{r,F}^{u,v} = \int_{B_0} e^{-rs} g^{u,v}(s)\, ds$$

# Obtaining solutions



- **Fredholm type equations**

$$v_\tau(x,y) = c_\tau(x,y) + \int_{A_0} v_\tau(u,w)\, f_\tau^{x,y}(u)\, du\, dw$$

- **discretized version** $\mathbf{v} = \mathbf{c} + \mathbf{M}\mathbf{v}$

- **solution needs care because of singularities in kernel**

# Maximum Process

- **joint density of the process and its maximum is**

$$f_\tau^u(x,y)=\frac{2(2y-x-u)}{\sqrt{2\pi\sigma^6\tau^3}}\exp\left\{-\frac{(x-u-\mu\tau)^2}{2\sigma^2\tau}\right\}\exp\left\{-\frac{2(y-u)(y-x)}{\sigma^2\tau}\right\}$$

- **inverse Gaussian hitting time distribution.**

- **The kernel**

$$K_\tau^u(x)=\frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left\{-\frac{(x-u-\mu\tau)^2}{2\sigma^2\tau}\right\}\times\left[1-\exp\left\{-\frac{2(s_n-u)(s_n-x)}{\sigma^2\tau}\right\}\right]$$

# The Integrated Process

$$Y_t = \int_{0 \le s \le t} X_s \, ds$$

- **basic process: drift μ and volatility** σ
- **Integrated process**

$$\mathbf{E}\left[Y_t\right] = \frac{1}{2}\mu t^2 \qquad \mathbf{V}\left[Y_t\right] = \frac{1}{3}\sigma^2 t^3$$

- **the joint density**

$$f_t^{0,0}(x,y) = \frac{\sqrt{3}}{\pi\sigma^2 t^2}\exp\left(-6\frac{(y-\frac{1}{2}\mu t^2)^2}{\sigma^2 t^3} + 6\frac{(x-\mu t)(y-\frac{1}{2}\mu t^2)}{\sigma^2 t^2} - 2\frac{(x-\mu t)^2}{\sigma^2 t}\right)$$

# Imperfect Inspection

- **degradation is observable with error** $Y_t = X_t + \varepsilon_t$
- **process is Gamma** $X_{t_j} - X_{t_i} \sim Ga(\alpha(t_j - t_i), \beta)$
- **error** $\varepsilon_t \sim N(0, \nu^2) \quad \forall\, t$
- **transition density**

$$f_{Y_t, X_t | X_0} = f_{Y_t | X_0, X_t} \, f_{X_t | X_0} = f_{Y_t | X_t} \, f_{X_t | X_0}$$

$$f_\tau^u(y, x) = \frac{1}{\nu\sqrt{2\pi}} \exp\left\{ -\frac{1}{2\nu^2}(y - x)^2 \right\} \frac{\beta^{\alpha\tau}(x-u)^{\alpha\tau - 1} \exp\{-\beta(x-u)\}}{\Gamma(\alpha\tau)}$$

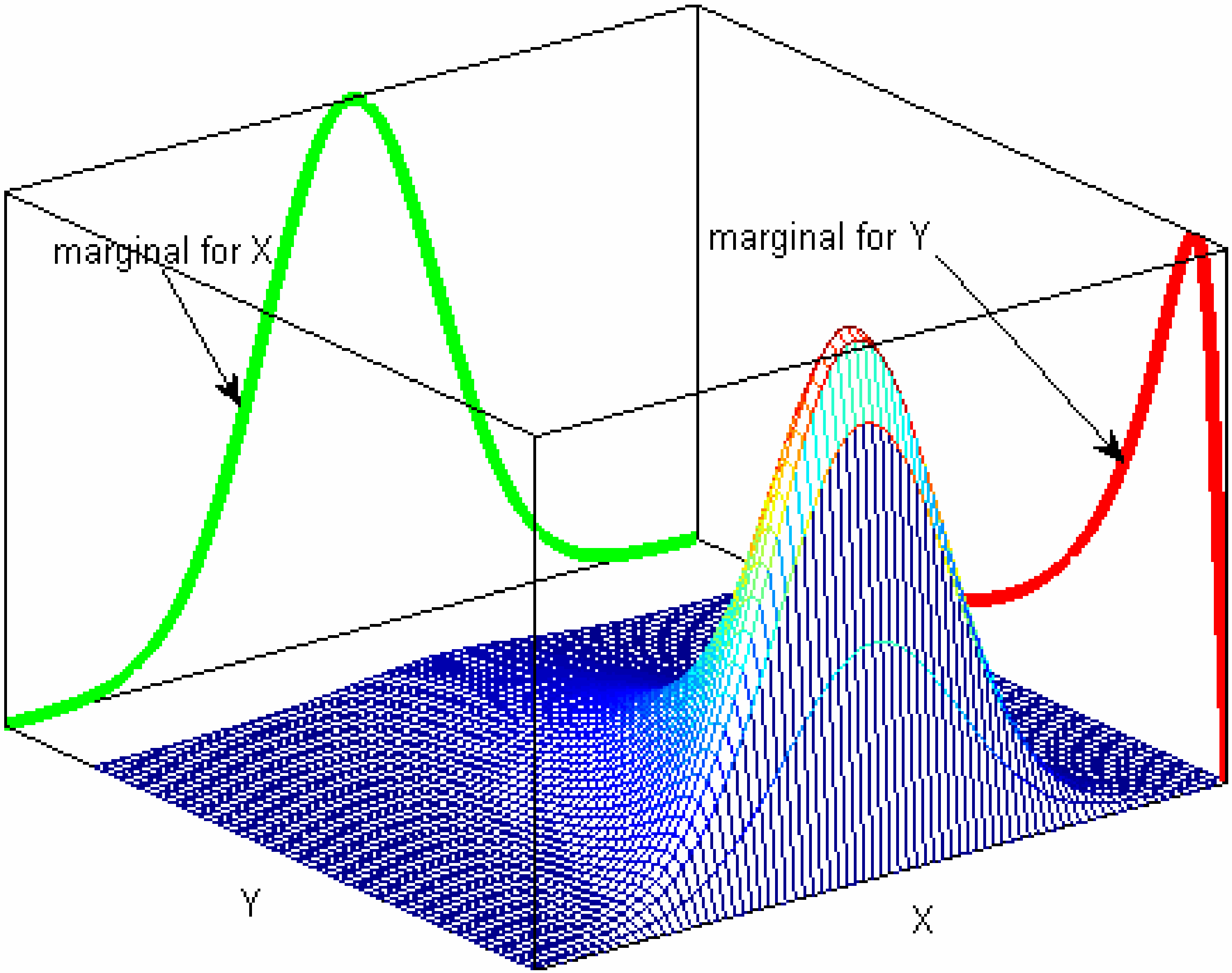
# summary

- **Common structure**
  - **Underlying process** $X_t$
  - **Bivariate process** $(X_t, Y_t)$
  - **Transition density** $f^{u,v}_{X_t,Y_t}(x,y)$
  - **Optimisation** $v_\tau(x,y) = \inf_{\tau>0} \left\{ c_\tau(x,y) + \int_{A_0} v_\tau(u,w)\, f_\tau^{x,y}(u,w)\, du\, dw \right\}$

# multivariate processes

- **hard to determine the dependence structure of mv process**
- **easy to observe marginal processes (one variable at a time)**
- **correlaton is not good enough**
- **use copulas**
- **the process is modelled in two parts**
  - the marginal distributions
  - the copula describes dependence
- **advantages**
  - marginals are observable
  - bounds on copulas are easily determined
  - copulas for transformed variables are readily obtained

# marginals



# copulas

**joint distribution** $F_{\underline{X}}(\underline{x}) = F_{\underline{X}}\left(x_1, x_2, \ldots, x_n\right)$

**we can observe the marginals** $F_{X1}(x_1), F_{X_2}(x_2), \cdots, F_{X_n}(x_n)$

**define** $C$ **by** $F_{\underline{X}}(\underline{x}) = C\left[F_{X1}(x_1), F_{X_2}(x_2), \cdots, F_{X_n}(x_n)\right]$

**copula is distribution on unit cube**

$$C:[0,1]^n \to [0,1]$$

**with uniform marginals**

$$C_{U_i}(u) = u$$

Sim-Serv Max-Serv TU/e 

# distributional bounds

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## Frechet bounds

$$\max\left\{\sum_{i=1}^{i=n} F_{X_i}(x_i)-(n-1),0\right\}\le F_{\underline{X}}(x_{1,}x_2\cdots,x_n)\le\min_{i=1..n}\left\{F_{X_i}(x_i)\right\}$$

$$\max\left\{\sum_{i=1}^{i=n} u_i-(n-1),0\right\}\le C(u_{1,}u_2\cdots,u_n)\le\min_{i=1..n}\left\{u_i\right\}$$

## Denuit bounds

$$\max\left\{\sum_{i=1}^{i=n} u_i-(n-1),0\right\}\le\cdots$$

$$C_{\theta_0}(u_{1,}u_2\cdots,u_n)\le C_{\theta}(u_{1,}u_2\cdots,u_n)\le C_{\theta_1}(u_{1,}u_2\cdots,u_n)\cdots$$

$$\le\min_{i=1..n}\left\{u_i\right\}$$

# distribution of decision variable

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**transformed variables**

$$Y = h(\underline{X})$$

$$Z_2 = X_2$$

$$\cdots$$

$$Z_n = X_n$$

**inverse**

$$X_1 = g(\underline{Y, Z})$$

$$X_2 = Z_2$$

$$\cdots$$

$$X_n = Z_n$$

**generalized convolutions**

$$\tau_{C,h}\left(F_{X_1}, F_{X_2}, \cdots, F_{X_n}\right)(y) = \sup_{z_2,\cdots,z_n} C\left\{F_{X_1}\left(g(y, z_2, \cdots z_n)\right), F_{X_2}(z_2), \cdots, F_{X_n}(z_n)\right\}$$

$$\rho_{C,h}\left(F_{X_1}, F_{X_2}, \cdots, F_{X_n}\right)(y) = \inf_{z_2,\cdots,z_n} C^d\left\{F_{X_1}\left(g(y, z_2, \cdots z_n)\right), F_{X_2}(z_2), \cdots, F_{X_n}(z_n)\right\}$$

# general bounds

**define**

$$F_0(y) = \tau_{C_0,h}\left(F_{X_1}, F_{X_2}, \cdots, F_{X_n}\right)(y)$$

$$F_1(y) = \rho_{C_1,h}\left(F_{X_1}, F_{X_2}, \cdots, F_{X_n}\right)(y)$$

**bounds are**

$$F_0(y) \le F_{h(X)}(y) \le F_1(y)$$

**percentiles**

$$F_1^{-1}(\alpha) \le F_{h(X)}^{-1}(\alpha) \le F_0^{-1}(\alpha)$$

# summary

- with copulas we can handle dependence without complete knowledge
- we get bounds on probabilities of interest

# future developments

- **development of effective performance criteria**
- **current process models are too simple yet difficult**
  - improve process models
  - solve decision problems
- **develop the model of interventions** $d(x,y)$
- **address multivariate process models using copulas**